جامعة بيرزيت

BIRZEIT UNIVERSITY

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Control Systems

Chapter 6: Stability

Introduction

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t) \quad (6.1)$$

Using these concepts, we present the following definitions of stability, instability, and marginal stability:

A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.

A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity.

A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

Introduction

Let us summarize our definitions of stability for linear, time-invariant systems. Using the natural response:

1. A system is stable if the natural response approaches zero as time approaches infinity.
2. A system is unstable if the natural response approaches infinity as time approaches infinity.
3. A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

Using the total response (BIBO):

1. A system is stable if *every* bounded input yields a bounded output.
2. A system is unstable if *any* bounded input yields an unbounded output.

Introduction

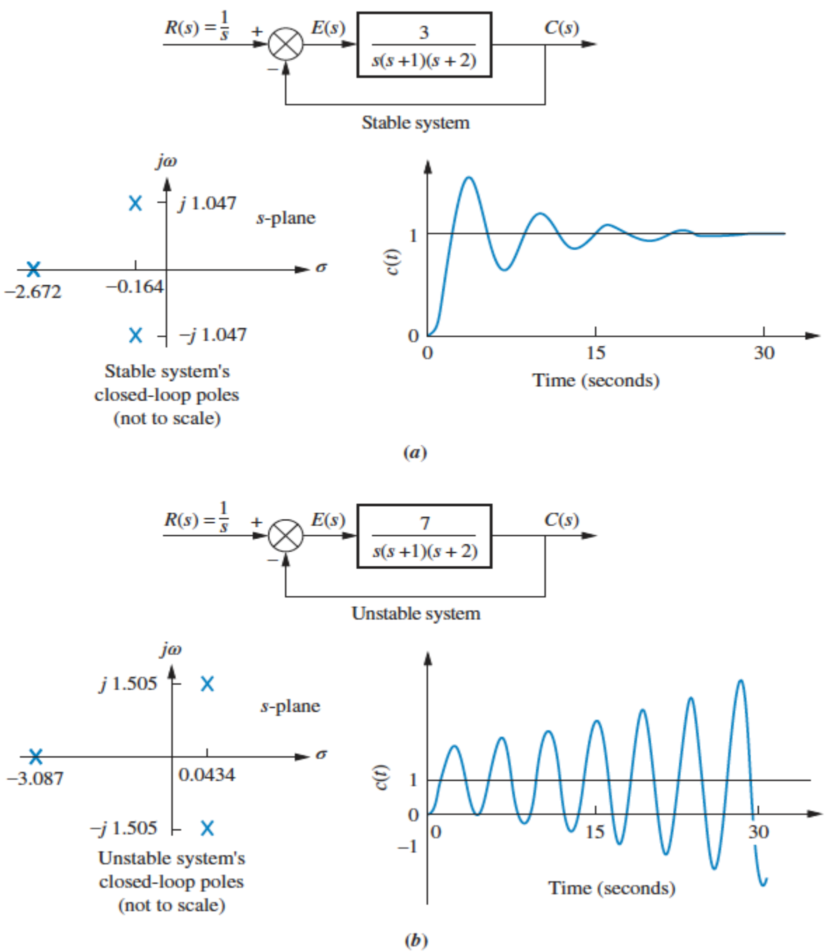

FIGURE 6.1 Closed-loop poles and response: **a.** stable system; **b.** unstable system

6.2 Routh-Hurwitz Criterion

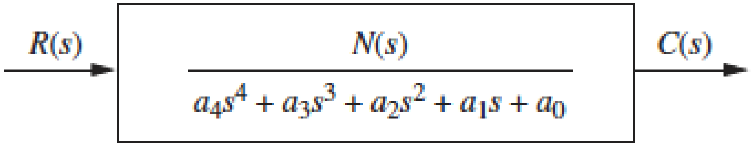

FIGURE 6.3 Equivalent closed-loop transfer function

TABLE 6.1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

6.2 Routh-Hurwitz Criterion

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

6.2 Routh-Hurwitz Criterion

Example 6.1

Creating a Routh Table

PROBLEM: Make the Routh table for the system shown in Figure 6.4(*a*).

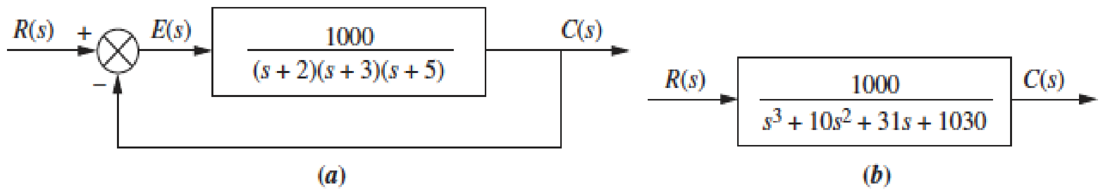

FIGURE 6.4 **a.** Feedback system for Example 6.1; **b.** equivalent closed-loop system

TABLE 6.3 Completed Routh table for Example 6.1

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

6.3 Routh-Hurwitz Criterion: Special Cases

Two special cases can occur: (1) The Routh table sometimes will have a zero *only in the first column* of a row, or (2) the Routh table sometimes will have an *entire row* that consists of zeros. Let us examine the first case.

Zero Only in the First Column

Example 6.2

Stability via Epsilon Method

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \tag{6.2}$$

SOLUTION:

TABLE 6.4 Completed Routh table for Example 6.2

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

TABLE 6.5 Determining signs in first column of a Routh table with zero as first element in a row

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

6.3 Routh-Hurwitz Criterion: Special Cases

Another method that can be used when a zero appears only in the first column of a row is derived from the fact that a polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same—right half-plane, left half-plane, or imaginary

Assume the equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{6.3}$$

If s is replaced by $1/d$, then d will have roots which are the reciprocal of s. Making this substitution in Eq. (6.3),

$$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \cdots + a_1\left(\frac{1}{d}\right) + a_0 = 0 \tag{6.4}$$

Factoring out $(1/d)^n$,

$$\left(\frac{1}{d}\right)^n \left[1 + a_{n-1}\left(\frac{1}{d}\right)^{-1} + \cdots + a_1\left(\frac{1}{d}\right)^{(1-n)} + a_0\left(\frac{1}{d}\right)^{-n}\right]$$

$$= \left(\frac{1}{d}\right)^n [1 + a_{n-1}d + \cdots + a_1 d^{(n-1)} + a_0 d^n] = 0 \tag{6.5}$$

6.3 Routh-Hurwitz Criterion: Special Cases

Example 6.3

Stability via Reverse Coefficients

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \quad (6.6)$$

SOLUTION: First write a polynomial that has the reciprocal roots of the denominator of Eq. (6.6). From our discussion, this polynomial is formed by writing the denominator of Eq. (6.6) in reverse order. Hence,

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1 \quad (6.7)$$

TABLE 6.6 Routh table for Example 6.3

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

6.3 Routh-Hurwitz Criterion: Special Cases

Entire Row is Zero

Example 6.4

Stability via Routh Table with Row of Zeros

PROBLEM: Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \qquad (6.8)$$

TABLE 6.7 Routh table for Example 6.4

s^5			1			6			8
s^4		~~7~~	1		~~42~~	6		~~56~~	8
s^3	~~0~~	~~4~~	1	~~0~~	~~12~~	3	~~0~~	~~0~~	0
s^2			3			8			0
s^1			$\frac{1}{3}$			0			0
s^0			8			0			0

$$P(s) = s^4 + 6s^2 + 8 \qquad (6.9)$$

Next we differentiate the polynomial with respect to s and obtain

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \qquad (6.10)$$

6.3 Routh-Hurwitz Criterion: Special Cases

Example 6.5

Pole Distribution via Routh Table with Row of Zeros

PROBLEM: For the transfer function

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20} \qquad (6.11)$$

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.

TABLE 6.8 Routh table for Example 6.5

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	−~~10~~ −1	−~~20~~ −2	~~10~~ 1	~~20~~ 2	0
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0
s^4	1	3	2	0	0
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0
s^2	~~$\frac{3}{2}$~~ 3	~~2~~ 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

$$P(s) = s^4 + 3s^2 + 2 \qquad (6.12)$$

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0 \qquad (6.13)$$

TABLE 6.9 Summary of pole locations for Example 6.5

	Polynomial		
Location	Even (fourth-order)	Other (fourth-order)	Total (eighth-order)
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

6.4 Routh-Hurwitz Criterion: Additional Examples

Example 6.6

Standard Routh-Hurwitz

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.6.

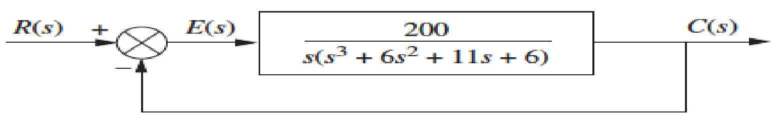

FIGURE 6.6 Feedback control system for Example 6.6

SOLUTION: First, find the closed-loop transfer function as

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200} \tag{6.14}$$

The Routh table for the denominator of Eq. (6.14) is shown as Table 6.10. For clarity, we leave most zero cells blank. At the s^1 row there is a negative coefficient; thus, there are two sign changes. The system is unstable, since it has two right-half-plane poles and two left-half-plane poles. The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.

TABLE 6.10 Routh table for Example 6.6

s^4		1		11	200
s^3	~~6~~	1	~~6~~	1	
s^2	~~10~~	1	~~200~~	20	
s^1		−19			
s^0		20			

6.4 Routh-Hurwitz Criterion: Additional Examples

Example 6.7

Routh-Hurwitz with Zero in First Column

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.7.

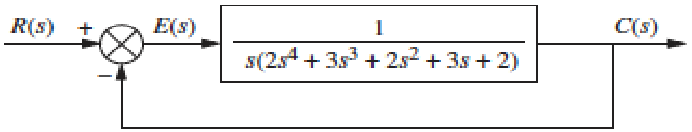

FIGURE 6.7 Feedback control system for Example 6.7

SOLUTION: The closed-loop transfer function is

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1} \quad (6.15)$$

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2 \quad (6.16)$$

TABLE 6.11 Routh table for Example 6.7

s^5	2	2	2
s^4	3	3	1
s^3	~~0~~ ϵ	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

6.4 Routh-Hurwitz Criterion: Additional Examples

We also can use the alternative approach, where we produce a polynomial whose roots are the reciprocal of the original. Using the denominator of Eq. (6.15), we form a polynomial by writing the coefficients in reverse order,

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2 \tag{6.16}$$

The Routh table for this polynomial is shown as Table 6.12. Unfortunately, in this case we also produce a zero only in the first column at the s^2 row. However, the table is easier to work with than Table 6.11. Table 6.12 yields the same results as Table 6.11: three poles in the left half-plane and two poles in the right-half-plane. The system is unstable.

TABLE 6.12 Alternative Routh table for Example 6.7

s^5	1	3	3
s^4	2	2	2
s^3	2	2	
s^2	~~0~~ ϵ	2	
s^1	$\frac{2\epsilon - 4}{\epsilon}$		
s^0	2		

6.4 Routh-Hurwitz Criterion: Additional Examples

Example 6.8

Routh-Hurwitz with Row of Zeros

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.8. Draw conclusions about the stability of the closed-loop system.

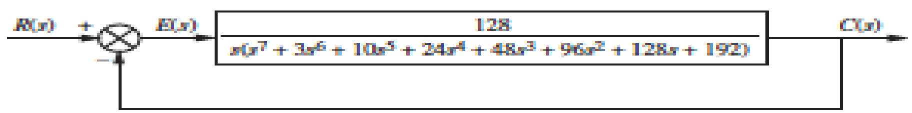

FIGURE 6.8 Feedback control system for Example 6.8

TryIt 6.2

Use MATLAB, The Control System Toolbox, and the following statements to find the closed-loop transfer function, $T(s)$, for Figure 6.8 and the closed-loop poles.

```
numg=128;
deng=[1 3 10 24 ...
  48 96 128 192 0];
G=tf(numg,deng);
T=feedback(G,1)
poles=pole(T)
```

SOLUTION: The closed-loop transfer function for the system of Figure 6.8 is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128} \qquad (6.17)$$

Using the denominator, form the Routh table shown as Table 6.13. A row of zeros appears in the s^5 row. Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64 \qquad (6.18)$$

TABLE 6.13 Routh table for Example 6.8

s^8			1			10			48			128	128
s^7		~~3~~	1		~~24~~	8		~~96~~	32		~~192~~	64	
s^6		~~2~~	1		~~16~~	8		~~64~~	32		~~128~~	64	
s^5	~~0~~	~~6~~	3	~~0~~	~~32~~	16	~~0~~	~~64~~	32	~~0~~	~~0~~	0	
s^4		~~$\frac{8}{3}$~~	1		~~$\frac{64}{3}$~~	8		~~64~~	24				
s^3		~~−8~~	−1		~~−40~~	−5							
s^2		~~3~~	1		~~24~~	8							
s^1			3										
s^0			8										

Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros:

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s + 0 \qquad (6.19)$$

6.4 Routh-Hurwitz Criterion: Additional Examples

poles. Since the even polynomial is of sixth order, the two remaining poles must be on the $j\omega$-axis.

There are no sign changes from the beginning of the table down to the even polynomial at the s^6 row. Therefore, the rest of the polynomial has no right-half-plane poles. The results are summarized in Table 6.14. The system has two poles in the right half-plane, four poles in the left half-plane, and two poles on the $j\omega$-axis, which are of unit multiplicity. The closed-loop system is unstable because of the right-half-plane poles.

TABLE 6.14 Summary of pole locations for Example 6.8

	Polynomial		
Location	**Even (sixth-order)**	**Other (second-order)**	**Total (eighth-order)**
Right half-plane	2	0	2
Left half-plane	2	2	4
$j\omega$	2	0	2

6.4 Routh-Hurwitz Criterion: Additional Examples

Example 6.9

Stability Design via Routh-Hurwitz

PROBLEM: Find the range of gain, K, for the system of Figure 6.10 that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.

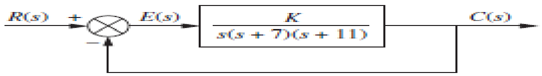

FIGURE 6.10 Feedback control system for Example 6.9

SOLUTION: First find the closed-loop transfer function as

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K} \tag{6.20}$$

Next form the Routh table shown as Table 6.15.

TABLE 6.15 Routh table for Example 6.9

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Since K is assumed positive, we see that all elements in the first column are always positive except the s^1 row. This entry can be positive, zero, or negative, depending upon the value of K. If $K < 1386$, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles in the left half-plane and be *stable*.

If $K > 1386$, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right-half-plane poles and one left-half-plane pole, which makes the system *unstable*.

If $K = 1386$, we have an entire row of zeros, which could signify $j\omega$ poles. Returning to the s^2 row and replacing K with 1386, we form the even polynomial

$$P(s) = 18s^2 + 1386 \tag{6.21}$$

Differentiating with respect to s, we have

$$\frac{dP(s)}{ds} = 36s + 0 \tag{6.22}$$

Replacing the row of zeros with the coefficients of Eq. (6.22), we obtain the Routh-Hurwitz table shown as Table 6.16 for the case of $K = 1386$.

TABLE 6.16 Routh table for Example 6.9 with $K = 1386$

s^3	1	77
s^2	18	1386
s^1	~~0~~ 36	
s^0	1386	

6.4 Routh-Hurwitz Criterion: Additional Examples

Example 6.10

Factoring via Routh-Hurwitz

PROBLEM: Factor the polynomial

$$s^4 + 3s^3 + 30s^2 + 30s + 200 \qquad (6.23)$$

SOLUTION: Form the Routh table of Table 6.17. We find that the s^1 row is a row of zeros. Now form the even polynomial at the s^2 row:

$$P(s) = s^2 + 10 \qquad (6.24)$$

TABLE 6.17 Routh table for Example 6.10

s^4	1	30	200
s^3	~~3~~ 1	~~30~~ 10	
s^2	~~20~~ 1	~~200~~ 10	
s^1	~~0~~ 2	~~0~~ 0	
s^0	10		

This polynomial is differentiated with respect to s in order to complete the Routh table. However, since this polynomial is a factor of the original polynomial in Eq. (6.23), dividing Eq. (6.23) by (6.24) yields $(s^2 + 3s + 20)$ as the other factor. Hence,

$$\begin{aligned} s^4 + 3s^3 + 30s^2 + 30s + 200 &= (s^2 + 10)(s^2 + 3s + 20) \\ &= (s + j3.1623)(s - j3.1623) \\ &\quad \times (s + 1.5 + j4.213)(s + 1.5 - j4.213) \qquad (6.25) \end{aligned}$$

6.4 Routh-Hurwitz Criterion: Additional Examples

Skill-Assessment Exercise 6.3

PROBLEM: For a unity feedback system with the forward transfer function

$$G(s) = \frac{K(s+20)}{s(s+2)(s+3)}$$

find the range of K to make the system stable.

ANSWER: $0 < K < 2$

The complete solution is at www.wiley.com/college/nise.